

## A NON-COMMUTATIVE WIENER-WINTNER THEOREM

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ABSTRACT. For a von Neumann algebra  $\mathcal{M}$  with a faithful normal tracial state  $\tau$  and a positive ergodic homomorphism  $\alpha : \mathcal{L}^1(\mathcal{M}, \tau) \rightarrow \mathcal{L}^1(\mathcal{M}, \tau)$  such that  $\tau \circ \alpha = \tau$  and  $\alpha$  does not increase the norm in  $\mathcal{M}$ , we establish a non-commutative counterpart of the classical Wiener-Wintner ergodic theorem.

## 1. INTRODUCTION AND PRELIMINARIES

The celebrated Wiener-Wintner theorem is by far one of the most deep and fruitful results of the classical ergodic theory. It may be stated as follows.

**Theorem 1.1.** *Let  $(\Omega, \mu)$  be a probability space, and let  $T : \Omega \rightarrow \Omega$  be an ergodic measure preserving transformation. Then for any function  $f \in L^1(\Omega, \mu)$  there exists a set  $\Omega_f$  of full measure such that, given  $\omega \in \Omega_f$ , the averages*

$$a_n(f, \lambda)(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k f(T^k \omega)$$

converge for all  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$ .

The aim of this article is to establish a non-commutative extension of Theorem 1.1. We follow the path of "simple inequality" as it is outlined in [1]. This means that our argument relies on a non-commutative Van der Corput's inequality. Note that such an inequality was established in [9].

Let  $H$  be a Hilbert space,  $B(H)$  the algebra of all bounded linear operators in  $H$ ,  $\|\cdot\|_\infty$  the uniform norm in  $B(H)$ ,  $\mathbb{I}$  the unit of  $B(H)$ . Let  $\mathcal{M} \subset B(H)$  be a semifinite von Neumann algebra with a faithful normal semifinite trace  $\tau$ . We denote by  $P(\mathcal{M})$  the complete lattice of all projections in  $\mathcal{M}$  and set  $e^\perp = \mathbb{I} - e$  whenever  $e \in P(\mathcal{M})$ .

A densely-defined closed operator  $x$  in  $H$  is said to be *affiliated* with the algebra  $\mathcal{M}$  if  $x'x \subset xx'$  for every  $x' \in B(H)$  such that  $x'x = xx'$  for all  $x \in \mathcal{M}$ . An operator  $x$  affiliated with  $\mathcal{M}$  is called  $\tau$ -*measurable* if for each  $\epsilon > 0$  there exists such  $e \in P(\mathcal{M})$  with  $\tau(e^\perp) \leq \epsilon$  that the subspace  $eH$  belongs to the domain of  $x$ . (In this case  $xe \in \mathcal{M}$ .) Let  $\mathcal{L} = \mathcal{L}(\mathcal{M}, \tau)$  be the set of all  $\tau$ -measurable operators affiliated with the algebra  $\mathcal{M}$ . The topology  $t_\tau$  defined in  $\mathcal{L}$  by the family

$V(\epsilon, \delta) = \{x \in \mathcal{L} : \|xe\|_\infty \leq \delta \text{ for some } e \in P(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon\}; \epsilon > 0, \delta > 0$   
of (closed) neighborhoods of zero is called a *measure topology*.

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**Theorem 1.2.** ([10]; see also [8])  $(\mathcal{L}, t_\tau)$  is a complete metrizable topological  $*$ -algebra.

For a positive self-adjoint operator  $x = \int_0^\infty \lambda de_\lambda$  affiliated with  $\mathcal{M}$  one can define

$$\tau(x) = \sup_n \tau \left( \int_0^n \lambda de_\lambda \right) = \int_0^\infty \lambda d\tau(e_\lambda).$$

If  $1 \leq p < \infty$ , then the non-commutative  $L^p$ -space associated with  $(\mathcal{M}, \tau)$  is defined as

$$\mathcal{L}^p = \mathcal{L}^p(\mathcal{M}, \tau) = \{x \in \mathcal{L} : \|x\|_p = (\tau(|x|^p))^{1/p} < \infty\},$$

where  $|x| = (x^*x)^{1/2}$ , the absolute value of  $x$ . Naturally,  $\mathcal{L}^\infty = \mathcal{M}$ .

Let  $\alpha : \mathcal{L}^1 \rightarrow \mathcal{L}^1$  be a positive linear map such that  $\alpha(x) \leq \mathbb{I}$  and  $\tau(\alpha(x)) \leq \tau(x)$  for every  $x \in \mathcal{L}^1 \cap \mathcal{M}$  with  $0 \leq x \leq \mathbb{I}$ . If  $\alpha$  is such a map, then, as is shown in [11, 12],  $\|\alpha(x)\|_p \leq \|x\|_p$  for each  $x = x^* \in \mathcal{L}^1 \cap \mathcal{M}$  and all  $1 \leq p \leq \infty$ . Besides, there exist unique continuous extensions  $\alpha : \mathcal{L}^p \rightarrow \mathcal{L}^p$  for every  $1 \leq p < \infty$  and a unique ultra-weakly continuous extension  $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ . This implies that for every  $1 \leq p \leq \infty$  and  $x \in \mathcal{L}^p$  we have  $\|\alpha(x)\|_p \leq 2\|x\|_p$ .

Let  $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$ . If  $1 \leq p \leq \infty$ ,  $x \in \mathcal{L}^p$ ,  $\lambda \in \mathbb{C}_1$ , we denote

$$(1) \quad a_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(x),$$

$$(2) \quad a_n(x, \lambda) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k \alpha^k(x).$$

There are several generally distinct types of "pointwise" (or "individual") convergence in  $\mathcal{L}$  each of which, in the commutative case with finite measure, reduces to the almost everywhere convergence. We deal with the so-called *almost uniform* (a.u.) and *bilateral almost uniform* (b.a.u.) convergences for which  $x_n \rightarrow x$  a.u. (b.a.u.) means that for every  $\epsilon > 0$  there exists such  $e \in P(\mathcal{M})$  that  $\tau(e^\perp) \leq \epsilon$  and  $\|(x - x_n)e\|_\infty \rightarrow 0$  ( $\|e(x - x_n)e\|_\infty \rightarrow 0$ , respectively). Clearly, a.u. convergence implies b.a.u. convergence.

In [12] the following non-commutative ergodic theorem was established.

**Theorem 1.3.** For every  $x \in \mathcal{L}^1$ , the ergodic averages (1) converge b.a.u. to some  $\hat{x} \in \mathcal{L}^1$ .

B.a.u. convergence of the averages (2) for  $x \in \mathcal{L}^1$  and a fixed  $\lambda \in \mathbb{C}_1$  was proved in [3].

## 2. NON-COMMUTATIVE WIENER-WINTNER PROPERTY

Now we turn our attention to a study of the "simultaneous" on  $\mathbb{C}_1$  individual convergence of the averages (2). We begin with the following definition; see [6].

**Definition 2.1.** Let  $(X, \|\cdot\|)$  be a normed space. A sequence  $a_n : X \rightarrow \mathcal{L}$  of additive maps is called *bilaterally uniformly equicontinuous in measure* (b.u.e.m.)

at  $0 \in X$  if for every  $\epsilon > 0$ ,  $\delta > 0$  there exists  $\gamma > 0$  such that for every  $x \in X$  with  $\|x\| < \gamma$  there is  $e_x \in P(\mathcal{M})$  for which

$$\tau(e_x^\perp) \leq \epsilon \quad \text{and} \quad \sup_n \|e_x a_n(x) e_x\|_\infty \leq \delta.$$

A proof of the following fact can be found in [6].

**Proposition 2.1.** *For any  $1 \leq p < \infty$  the sequence  $\{a_n\}$  given by (1) is b.u.e.m. at  $0 \in \mathcal{L}^p$ .*

**Lemma 2.1.** *If  $1 \leq p < \infty$ , then, given  $\epsilon > 0$ ,  $\delta > 0$ , there exists  $\gamma > 0$  such that for every  $x \in \mathcal{L}^p$  with  $\|x\|_p \leq \gamma$  there is  $e \in P(\mathcal{M})$  satisfying*

$$\tau(e^\perp) \leq \epsilon \quad \text{and} \quad \sup_n \|e a_n(x, \lambda) e\|_\infty \leq \delta \quad \text{for all } \lambda \in \mathbb{C}_1.$$

*Proof.* Fix  $\epsilon > 0$ ,  $\delta > 0$ . By Proposition 2.1, there exists  $\gamma > 0$  such that for each  $\|x\|_p < \gamma$  it is possible to find  $e \in P(\mathcal{M})$  such that

$$\tau(e^\perp) \leq \frac{\epsilon}{4} \quad \text{and} \quad \sup_n \|e a_n(x) e\|_\infty \leq \frac{\delta}{24}.$$

Fix  $x \in \mathcal{L}^p$  with  $\|x\|_p < \gamma$ . We have  $x = (x_1 - x_2) + i(x_3 - x_4)$ , where  $x_j \in \mathcal{L}_+^p$  and  $\|x_j\|_p \leq \|x\|_p$  for each  $j = 1, 2, 3, 4$ .

If  $1 \leq j \leq 4$ , then  $\|x_j\|_p < \gamma$ , so there is  $e_j \in P(\mathcal{M})$  satisfying

$$\tau(e_j^\perp) \leq \frac{\epsilon}{4} \quad \text{and} \quad \sup_n \|e_j a_n(x_j) e_j\|_\infty \leq \frac{\delta}{24}.$$

Let  $e = \wedge_{j=1}^4 e_j$ . Then we have

$$\tau(e^\perp) \leq \epsilon \quad \text{and} \quad \sup_n \|e a_n(x_j) e\|_\infty \leq \frac{\delta}{24}, \quad j = 1, 2, 3, 4.$$

Now, fix  $\lambda \in \mathbb{C}_1$ . For  $1 \leq j \leq 4$  denote

$$a_n^{(R)}(x_j, \lambda) = \frac{1}{n} \sum_{k=0}^{n-1} \operatorname{Re}(\lambda^k) \alpha^k(x_j) + a_n(x_j) = \frac{1}{n} \sum_{k=0}^{n-1} (\operatorname{Re}(\lambda^k) + 1) \alpha^k(x_j),$$

$$a_n^{(I)}(x_j, \lambda) = \frac{1}{n} \sum_{k=0}^{n-1} \operatorname{Im}(\lambda^k) \alpha^k(x_j) + a_n(x_j) = \frac{1}{n} \sum_{k=0}^{n-1} (\operatorname{Im}(\lambda^k) + 1) \alpha^k(x_j).$$

Then  $0 \leq \operatorname{Re}(\lambda^k) + 1 \leq 2$  and  $\alpha^k(x_j) \geq 0$  for every  $k$  entail

$$0 \leq e a_n^{(R)}(x_j, \lambda) e \leq 2 e a_n(x_j) e \quad \text{for all } n.$$

Therefore

$$\sup_n \|e a_n^{(R)}(x_j, \lambda) e\|_\infty \leq \frac{\delta}{12}$$

and, similarly,

$$\sup_n \|e a_n^{(I)}(x_j, \lambda) e\|_\infty \leq \frac{\delta}{12}.$$

This implies that, given  $1 \leq j \leq 4$ , we have

$$\begin{aligned} & \sup_n \|e a_n(x_j, \lambda) e\|_\infty = \\ & = \sup_n \left\| e \left( a_n^{(R)}(x_j, \lambda) + i a_n^{(I)}(x_j, \lambda) - a_n(x_j) - i a_n(x_j) \right) e \right\|_\infty \leq \frac{\delta}{4}, \end{aligned}$$

and we conclude that

$$\begin{aligned} & \sup_n \|ea_n(x, \lambda)e\|_\infty = \\ & = \sup_n \|e(a_n(x_1, \lambda) - a_n(x_2, \lambda) + ia_n(x_3, \lambda) - ia_n(x_4, \lambda))e\|_\infty \leq \delta \end{aligned}$$

for every  $\lambda \in \mathbb{C}_1$ .  $\square$

**Definition 2.2.** Let  $1 \leq p < \infty$ . We say that  $x \in \mathcal{L}^p$  satisfies *Wiener-Wintner (bilaterally Wiener-Wintner) property* and we write  $x \in WW$  ( $x \in bWW$ , respectively) if, given  $\epsilon > 0$ , there exists a projection  $p \in P(\mathcal{M})$  with  $\tau(p^\perp) \leq \epsilon$  such that the sequence

$$\{a_n(x, \lambda)p\} \quad (\{pa_n(x, \lambda)p\}, \text{ respectively}) \quad \text{converges in } \mathcal{M} \text{ for all } \lambda \in \mathbb{C}_1.$$

Note that  $WW \subset bWW$ , while in the commutative case these sets coincide.

Let  $(\Omega, \mu)$  be a probability space, and let  $T : \Omega \rightarrow \Omega$  be a measure preserving transformation. Then  $f \in L^1(\Omega, \mu) \cap WW$  would imply that for every  $m \in \mathbb{N}$  there exists  $\Omega_m$  with  $\mu(\Omega \setminus \Omega_m) \leq \frac{1}{m}$  such that the averages  $a_n(f, \lambda)(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k f(T^k \omega)$  converge for all  $\omega \in \Omega_m$  and  $\lambda \in \mathbb{C}_1$ . Then, with  $\Omega_f = \bigcup_{m=1}^\infty \Omega_m$ , we have  $\mu(\Omega_f) = 1$ , while the averages  $a_n(f, \lambda)(\omega)$  converge for all  $\omega \in \Omega_f$  and  $\lambda \in \mathbb{C}_1$ .

Therefore Definition 2.2 presents a proper generalization of the classical Wiener-Wintner property; see [1], p.28. In an attempt to clarify what happens in the non-commutative situation without imposing any additional conditions on  $\tau$  and  $\alpha$ , we suggest the following.

**Proposition 2.2.** Let  $1 \leq p < \infty$  and  $x \in \mathcal{L}^p \cap WW$  ( $x \in \mathcal{L}^p \cap bWW$ ). Then

(1) for every  $\lambda \in \mathbb{C}_1$  there is such  $x_\lambda \in \mathcal{L}^p$  that

$$a_n(x, \lambda) \rightarrow x_\lambda \text{ a.u.} \quad (a_n(x, \lambda) \rightarrow x_\lambda \text{ b.a.u., respectively}),$$

(2) if  $p \in P(\mathcal{M})$  is such that  $\{a_n(x, \lambda)p\} \quad (\{pa_n(x, \lambda)p\})$  converges in  $\mathcal{M}$  for all  $\lambda \in \mathbb{C}_1$ , then, given  $\lambda \in \mathbb{C}_1$  and  $\nu > 0$ , there is a projection  $p_\lambda \in P(\mathcal{M})$  such that  $p_\lambda \leq p$ ,  $\tau(p - p_\lambda) \leq \nu$ , and

$$\|(a_n(x, \lambda) - x_\lambda)p_\lambda\|_\infty \rightarrow 0 \quad (\|p_\lambda(a_n(x, \lambda) - x_\lambda)p_\lambda\|_\infty \rightarrow 0, \text{ respectively}).$$

*Proof.* We will provide a proof for the b.a.u. convergence. Same argument is applicable for the a.u. convergence.

(1) Let  $x \in \mathcal{L}^p \cap bWW$  and  $\lambda \in \mathbb{C}_1$ . Then for every  $\epsilon > 0$  there exists  $p \in P(\mathcal{M})$  with  $\tau(p^\perp) \leq \epsilon$  for which

$$\|p(a_m(x, \lambda) - a_n(x, \lambda))p\|_\infty \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Then, as it is noticed in [2], Proposition 1.3,  $a_n(x, \lambda) \rightarrow x_\lambda$  b.a.u. for some  $x_\lambda \in \mathcal{L}$ , which clearly implies that  $a_n(x, \lambda) \rightarrow x_\lambda$  *bilaterally in measure*, meaning that, given  $\epsilon > 0$ ,  $\delta > 0$ , there exists  $N \in \mathbb{N}$  such that for every  $n \geq N$  there is  $e_n \in P(\mathcal{M})$  with  $\tau(e_n^\perp) \leq \epsilon$  satisfying  $\|e_n(a_n(x, \lambda) - x_\lambda)e_n\|_\infty \leq \delta$ . Since the measure topology coincides with the bilateral measure topology on  $\mathcal{L}$  ([3], Theorem 2.2), we have  $a_n(x, \lambda) \rightarrow x_\lambda$  in measure. Then, since  $\|a_n(x, \lambda)\|_p \leq 2\|x\|_p$  for every  $n$ , by Theorem 1.2 in [3],  $x_\lambda \in \mathcal{L}^p$ .

(2) Let  $p \in P(\mathcal{M})$  be such that the sequence  $\{pa_n(x, \lambda)p\}$  converges in  $\mathcal{M}$  for all  $\lambda \in \mathbb{C}_1$ . By (1), given  $\lambda \in \mathbb{C}_1$  and  $\nu > 0$ , there is  $e_\lambda \in P(\mathcal{M})$  with  $\tau(e_\lambda^\perp) \leq \nu$  such that  $\|e_\lambda(a_n(x, \lambda) - x_\lambda)e_\lambda\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $p_\lambda = p \wedge e_\lambda$  satisfies the required conditions.  $\square$

**Remark 2.1.** It is desirable to have the following: if  $x \in WW$  ( $x \in bWW$ ), then, given  $\epsilon > 0$ , there exists such  $p \in P(\mathcal{M})$  with  $\tau(p^\perp) \leq \epsilon$  that  $\|(a_n(x, \lambda) - x_\lambda)p\|_\infty \rightarrow 0$  ( $\|p(a_n(x, \lambda) - x_\lambda)p\|_\infty \rightarrow 0$ , respectively) for all  $\lambda \in \mathbb{C}_1$ ; see Remark 5.1 below.

**Theorem 2.1.** *For each  $1 \leq p < \infty$  the set  $X = \mathcal{L}^p \cap bWW$  is closed in  $\mathcal{L}^p$ .*

*Proof.* Take  $x$  in the  $\|\cdot\|_p$ -closure of  $X$  and fix  $\epsilon > 0$ . By Lemma 2.1, one can find sequences  $\{x_m\} \subset X$  and  $\{e_m\} \subset P(\mathcal{M})$  in such a way that

$$\tau(e_m^\perp) \leq \frac{\epsilon}{3 \cdot 2^m} \quad \text{and} \quad \sup_n \|e_m a_n(x - x_m, \lambda) e_m\|_\infty \leq \frac{1}{m}$$

for all  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C}_1$ . If we let  $e = \bigwedge_{m=1}^\infty e_m$ , then

$$\tau(e^\perp) \leq \frac{\epsilon}{3} \quad \text{and} \quad \sup_n \|e a_n(x - x_m, \lambda) e\|_\infty \leq \frac{1}{m},$$

$m \in \mathbb{N}$ ,  $\lambda \in \mathbb{C}_1$ . Also, since  $\{x_m\} \subset bWW$ , one can construct  $f \in P(\mathcal{M})$  such that

$$\tau(f^\perp) \leq \frac{\epsilon}{3} \quad \text{and} \quad \{f a_n(x_m, \lambda) f\} \text{ converges in } \mathcal{M} \text{ for all } m \in \mathbb{N}, \lambda \in \mathbb{C}_1.$$

Next, there exists  $g \in P(\mathcal{M})$  with  $\tau(g^\perp) \leq \frac{\epsilon}{3}$  for which  $\{\alpha^k(x)g\}_{k=0}^\infty \subset \mathcal{M}$  so that  $\{g a_n(x, \lambda) g\} \subset \mathcal{M}$  for all  $\lambda \in \mathbb{C}_1$ . Now, if  $p = e \wedge f \wedge g$ , then we have  $\tau(p^\perp) \leq \epsilon$ ,

$$\sup_n \|p a_n(x - x_m, \lambda) p\|_\infty \leq \frac{1}{m},$$

$$\{p a_n(x_m, \lambda) p\} \text{ converges in } \mathcal{M},$$

$$\text{and } \{p a_n(x, \lambda) p\} \subset \mathcal{M}$$

for all  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C}_1$ .

It remains to show that, for a fixed  $\lambda \in \mathbb{C}_1$ , the sequence  $\{p a_n(x, \lambda) p\}$  converges in  $\mathcal{M}$ . So, fix  $\delta > 0$  and pick  $m_0$  such that  $\frac{1}{m_0} \leq \frac{\delta}{3}$ . Since the sequence  $\{p a_n(x_{m_0}, \lambda) p\}$  converges in  $\mathcal{M}$ , there exists  $N$  such that

$$\|p(a_{n_1}(x_{m_0}, \lambda) - a_{n_2}(x_{m_0}, \lambda))p\|_\infty \leq \frac{\delta}{3}$$

whenever  $n_1, n_2 \geq N$ . Therefore, given  $n_1, n_2 \geq N$ , we can write

$$\begin{aligned} \|p(a_{n_1}(x, \lambda) - a_{n_2}(x, \lambda))p\|_\infty &\leq \|p a_{n_1}(x - x_{m_0}, \lambda) p\|_\infty + \\ &+ \|p a_{n_2}(x - x_{m_0}, \lambda) p\|_\infty + \|p(a_{n_1}(x_{m_0}, \lambda) - a_{n_2}(x_{m_0}, \lambda))p\|_\infty \leq \delta. \end{aligned}$$

This implies that the sequence  $\{p a_n(x, \lambda) p\}$  converges in  $\mathcal{M}$  for all  $\lambda \in \mathbb{C}_1$ , hence  $x \in X$  and  $X$  is closed in  $\mathcal{L}^p$ .  $\square$

Let  $\mathcal{K}$  be the  $\|\cdot\|_2$ -closure of the linear span of the set

$$(3) \quad E = \{x \in \mathcal{L}^2 : \alpha(x) = \mu x \text{ for some } \mu \in \mathbb{C}_1\}.$$

**Proposition 2.3.**  $\mathcal{K} \subset bWW$ .

*Proof.* By Theorem 2.1, it is sufficient to show that  $\sum_{j=1}^m a_j x_j \in bWW$  whenever  $a_j \in \mathbb{C}$  and  $x_j \in E$ ,  $1 \leq j \leq m$ . For this, one will verify that  $E \subset WW$ .

If  $x \in E$ , then  $\alpha(x) = \mu x$ ,  $\mu \in \mathbb{C}_1$ . Fix  $\epsilon > 0$  and find  $p \in P(\mathcal{M})$  with  $\tau(p^\perp) \leq \epsilon$  such that  $x p \in \mathcal{M}$ . Then, given  $\lambda \in \mathbb{C}_1$ , we have

$$a_n(x, \lambda) = x p \frac{1}{n} \sum_{k=0}^{n-1} (\lambda \mu)^k.$$

Therefore, since the averages  $\frac{1}{n} \sum_{k=0}^{n-1} (\lambda \mu)^k$  converge, we conclude that the sequence  $\{a_n(x, \lambda)p\}$  converges in  $\mathcal{M}$ , whence  $x \in WW$ .  $\square$

### 3. SPECTRAL CHARACTERIZATION OF $\mathcal{K}^\perp$

The space  $\mathcal{L}^2$  equipped with the inner product  $(x, y)_\tau = \tau(x^*y)$  is a Hilbert space such that  $\|x\|_2 = \|x\|_\tau = (x, x)_\tau^{1/2}$ ,  $x \in \mathcal{L}^2$ .

From now on we shall assume that  $\tau$  and  $\alpha$  satisfy the following additional conditions:  $\tau$  is a state,  $\alpha$  is a homomorphism, and  $\tau \circ \alpha = \tau$ . Notice that then  $\|\alpha(x)\|_2 = \|x\|_2$  and  $|\tau(x)| \leq \|x\|_2$  for every  $x \in \mathcal{L}^2$ .

**Proposition 3.1.** *If  $x \in \mathcal{L}^2$ , then the sequence  $\{\gamma_x(l)\}_{l=-\infty}^\infty$  given by*

$$\gamma_x(l) = \begin{cases} \tau(x^* \alpha^l(x)), & \text{if } l \geq 0 \\ \tau(x^* \alpha^{-l}(x)), & \text{if } l < 0. \end{cases}$$

*is positive definite.*

*Proof.* If  $\mu_0, \dots, \mu_m \in \mathbb{C}$ , then, taking into account that positivity of  $\alpha$  implies that  $\alpha(y)^* = \alpha(y^*)$ ,  $y \in \mathcal{L}^2$ , we have

$$0 \leq \left\| \sum_{k=0}^m \mu_k \alpha^k(x) \right\|_2^2 = \left( \sum_{j=0}^m \mu_j \alpha^j(x), \sum_{i=0}^m \mu_i \alpha^i(x) \right)_\tau = \sum_{i,j=0}^m \mu_i \bar{\mu}_j \tau(\alpha^j(x^*) \alpha^i(x)).$$

If  $i \geq j$ , we can write

$$\tau(\alpha^j(x^*) \alpha^i(x)) = \tau(\alpha^j(x^* \alpha^{i-j}(x))) = \tau(x^* \alpha^{i-j}(x)) = \gamma_x(i-j),$$

and if  $i < j$ , we have

$$\tau(\alpha^j(x^*) \alpha^i(x)) = \overline{\tau(\alpha^i(x^*) \alpha^j(x))} = \overline{\tau(x^* \alpha^{j-i}(x))} = \gamma_x(i-j).$$

Therefore

$$\sum_{i,j=0}^m \gamma_x(i-j) \mu_i \bar{\mu}_j \geq 0$$

for any  $\mu_0, \dots, \mu_m \in \mathbb{C}$ , hence  $\{\gamma_x(l)\}$  is positive definite.  $\square$

Consequently, given  $x \in \mathcal{L}^2$ , by Herglotz-Bochner theorem, there exists a positive finite Borel measure  $\sigma_x$  on  $\mathbb{C}_1$  such that

$$(4) \quad \tau(x^* \alpha^l(x)) = \gamma_x(l) = \widehat{\sigma_x}(l) = \int_{\mathbb{C}_1} e^{2\pi i l t} d\sigma_x(t), \quad l = 1, 2, \dots$$

**Lemma 3.1.**  $\alpha(\mathcal{K}^\perp) \subset \mathcal{K}^\perp$ .

*Proof.* Since  $\alpha : \mathcal{L}^2 \rightarrow \mathcal{L}^2$  is an isometry, we have  $\|\alpha\| = 1$ . Therefore  $\|\alpha^*\| = 1$  as well, so that  $\|\alpha^*(x)\|_2 \leq \|x\|_2$ ,  $x \in \mathcal{L}^2$ .

Let  $x \in E$ , that is,  $x \in \mathcal{L}^2$  and  $\alpha(x) = \mu x$  for some  $\mu \in \mathbb{C}_1$ . Then we have

$$\|\alpha^*(x) - \bar{\mu}x\|_2^2 = \|\alpha^*(x)\|_2^2 - \bar{\mu}(\alpha^*(x), x)_\tau - \mu(x, \alpha^*(x))_\tau + \|x\|_2^2 \leq 0,$$

and it follows that  $\alpha^*(x) = \bar{\mu}x$ .

Now, if  $y \in \mathcal{K}^\perp$ , then  $(\alpha(y), x)_\tau = (y, \alpha^*(x))_\tau = \bar{\mu}(y, x)_\tau = 0$ , which implies that  $\alpha(y) \perp E$ , hence  $\alpha(y) \in \mathcal{K}^\perp$ .  $\square$

**Proposition 3.2.** *If  $x \in \mathcal{K}^\perp$ , then the measure  $\sigma_x$  is continuous.*

*Proof.* We need to show that  $\sigma_x(\{t\}) = 0$  for every  $t \in \mathbb{C}_1$ . It is known ([5], p.42) that

$$\sigma_x(\{t\}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n e^{2\pi i l t} \widehat{\sigma_x}(t),$$

which is equal to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n e^{2\pi i l t} \tau(x^* \alpha^l(x)) = \lim_{n \rightarrow \infty} \tau \left( x^* \left( \frac{1}{n} \sum_{l=1}^n e^{2\pi i l t} \alpha^l(x) \right) \right).$$

Therefore it is sufficient to verify that

$$(5) \quad \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{l=1}^n e^{2\pi i l t} \alpha^l(x) \right\|_2 = 0.$$

By the Mean Ergodic theorem applied to  $\tilde{\alpha} : \mathcal{L}^2 \rightarrow \mathcal{L}^2$  given by  $\tilde{\alpha}(x) = e^{2\pi i t} \alpha(x)$ , we conclude that

$$\frac{1}{n} \sum_{l=1}^n e^{2\pi i l t} \alpha^l(x) \rightarrow \bar{x} \text{ in } \mathcal{L}^2.$$

Since  $x \in \mathcal{K}^\perp$ , by Lemma 3.1, we have  $\alpha^l(x) \in \mathcal{K}^\perp$  for each  $l$ , which implies that  $\bar{x} \in \mathcal{K}^\perp$ . Besides,

$$\alpha(\bar{x}) = \|\cdot\|_2 - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n e^{2\pi i l t} \alpha^{l+1}(x) = e^{-2\pi i t} \bar{x},$$

so that  $\bar{x} \in \mathcal{K}$ . Therefore  $\bar{x} = 0$ , and (5) follows.  $\square$

#### 4. NON-COMMUTATIVE VAN DER CORPUT'S INEQUALITY

It was shown in [9] that the extremely useful Van der Corput's "Fundamental Inequality" (see [1]) can be fully extended to any  $*$ -algebra:

**Theorem 4.1.** [9] *If  $n \geq 1$ ,  $0 \leq m \leq n-1$  are integers and  $a_0, \dots, a_{n-1}$  are elements of a  $*$ -algebra, then*

$$\left( \sum_{k=0}^{n-1} a_k^* \right) \left( \sum_{k=0}^{n-1} a_k \right) \leq \frac{n-1+m}{m+1} \sum_{k=0}^{n-1} a_k^* a_k + \frac{2(n-1+m)}{m+1} \sum_{l=1}^m \frac{m-l+1}{m+1} \operatorname{Re} \sum_{k=0}^{n-1} a_k^* a_{k+l}.$$

**Corollary 4.1.** *If in Theorem 4.1,  $a_0, \dots, a_{n-1}$  are elements of a  $C^*$ -algebra with the norm  $\|\cdot\|$ , then*

$$\left\| \sum_{k=0}^{n-1} a_k \right\|^2 \leq \frac{n-1+m}{m+1} \left\| \sum_{k=0}^{n-1} a_k^* a_k \right\| + \frac{2(n-1+m)}{m+1} \sum_{l=1}^m \frac{m-l+1}{m+1} \left\| \sum_{k=0}^{n-1} a_k^* a_{k+l} \right\|,$$

which implies that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} a_k \right\|^2 \leq \frac{n-1+m}{(m+1)n} \left\| \sum_{k=0}^{n-1} a_k^* a_k \right\| + \frac{2(n-1+m)}{(m+1)n} \sum_{l=1}^m \frac{m-l+1}{m+1} \left\| \frac{1}{n} \sum_{k=0}^{n-1} a_k^* a_{k+l} \right\|,$$

and further

$$(6) \quad \left\| \frac{1}{n} \sum_{k=0}^{n-1} a_k \right\|^2 < \frac{2}{m+1} \left\| \frac{1}{n} \sum_{k=0}^{n-1} a_k^* a_k \right\| + \frac{4}{m+1} \sum_{l=1}^m \left\| \frac{1}{n} \sum_{k=0}^{n-1} a_k^* a_{k+l} \right\|.$$

## 5. PROOF OF THE MAIN RESULT

We will assume now that  $\alpha$  is ergodic on  $\mathcal{L}^2$ , that is,  $\alpha(x) = x$ ,  $x \in \mathcal{L}^2$ , implies that  $x = c \cdot \mathbb{I}$ ,  $c \in \mathbb{C}$ .

**Proposition 5.1.** *If  $x \in \mathcal{L}^2$ , then  $a_n(x) \rightarrow \tau(x) \cdot \mathbb{I}$  a.u.*

*Proof.* By the Mean Ergodic theorem,  $a_n(x) \rightarrow \bar{x}$  in  $\mathcal{L}^2$ . Therefore  $\alpha(a_n(x)) \rightarrow \alpha(\bar{x})$  in  $\mathcal{L}^2$ , so  $\alpha(\bar{x}) = \bar{x}$ , and the ergodicity of  $\alpha$  implies that  $\bar{x} = c(x) \cdot \mathbb{I}$ . Then, since  $\tau$  is also continuous in  $\mathcal{L}^2$ , we have  $\tau(a_n(x)) \rightarrow \tau(\bar{x}) = c(x)$ , hence  $c(x) = \tau(x)$  because  $\tau(a_n(x)) = \tau(x)$  for each  $n$ . It is known ([4], [6]) that  $a_n(x) \rightarrow \hat{x} \in \mathcal{L}^2$  a.u., which implies that  $a_n(x) \rightarrow \hat{x}$  in measure. Since  $\|\cdot\|_2$ -convergence entails convergence in measure, we conclude that  $\hat{x} = \bar{x} = \tau(x) \cdot \mathbb{I}$ .  $\square$

**Lemma 5.1.** *If  $a, b \in \mathcal{L}$  and  $e \in P(\mathcal{M})$  are such that  $ae, be \in \mathcal{M}$ , then*

$$(ae)^*be = ea^*be.$$

*Proof.* We have

$$((ae)^*be)^* = (be)^*ae \subset (be)^*(ea^*)^* \subset (ea^*be)^*,$$

which, since  $((ae)^*be)^* \in B(H)$ , implies that  $((ae)^*be)^* = (ea^*be)^*$ , hence the required equality.  $\square$

Now we can prove our main result, a non-commutative Wiener-Wintner theorem.

**Theorem 5.1.** *Let  $\mathcal{M}$  be a von Neumann algebra,  $\tau$  a faithful normal tracial state on  $\mathcal{M}$ . Let  $\alpha : \mathcal{L}^1 \rightarrow \mathcal{L}^1$  be a positive ergodic homomorphism such that  $\tau \circ \alpha = \tau$  and  $\|\alpha(x)\|_\infty \leq \|x\|_\infty$ ,  $x \in \mathcal{M}$ . Then  $\mathcal{L}^1 = bWW$ , that is, for every  $x \in \mathcal{L}^1$  and  $\epsilon > 0$  there exists such a projection  $p \in P(\mathcal{M})$  that*

$$\tau(p^\perp) \leq \epsilon \quad \text{and} \quad \{pa_n(x, \lambda)p\} \text{ converges in } \mathcal{M} \text{ for all } \lambda \in \mathbb{C}_1.$$

*Proof.* Since  $\mathcal{L}^2$  is dense in  $\mathcal{L}^1$ ,  $\mathcal{L}^2 = \mathcal{K} \oplus \mathcal{K}^\perp$ , and  $\mathcal{K} \subset bWW$  (Proposition 2.3), by Theorem 2.1, it remains to show that  $\mathcal{K}^\perp \subset bWW$ . (In fact, we will show that  $\mathcal{K}^\perp \subset WW$ .)

So, let  $x \in \mathcal{K}^\perp$  and fix  $\epsilon > 0$ . Since  $\{x^*\alpha^l(x)\}_{l=0}^\infty \subset \mathcal{L}^2$ , due to Proposition 5.1, one can construct a projection  $p \in P(\mathcal{M})$  in such a way that

$$\tau(p^\perp) \leq \epsilon, \quad \{\alpha^k(x)p\} \subset \mathcal{M} \quad \text{for all } k,$$

$$pa_n(x^*x)p \rightarrow \tau(x^*x)p = \|x\|_2^2 p \quad \text{in } \mathcal{M}, \text{ and}$$

$$pa_n(x^*\alpha^l(x))p \rightarrow \tau(x^*\alpha^l(x))p = \widehat{\sigma_x}(l)p \quad \text{in } \mathcal{M} \text{ for every } l.$$

Now, if  $a_k = \lambda^k \alpha^k(x)p$ ,  $k = 0, 1, 2, \dots$ , then, employing Lemma 5.1, we obtain

$$a_k^*a_{k+l} = \lambda^l p \alpha^k(x^*\alpha^l(x))p, \quad k, l = 0, 1, 2, \dots$$

At this moment we apply inequality (6) to the sequence  $\{a_k\} \subset \mathcal{M}$  yielding in view of (1) and (2) that

$$\sup_{\lambda \in \mathbb{C}_1} \|a_n(x, \lambda)p\|_\infty^2 \leq \frac{2}{m+1} \|pa_n(x^*x)p\|_\infty + \frac{4}{m+1} \sum_{l=1}^m \|pa_n(x^*\alpha^l(x))p\|_\infty.$$

Therefore, for a fixed  $m$ , we have

$$\limsup_n \sup_{\lambda \in \mathbb{C}_1} \|a_n(x, \lambda)p\|_\infty^2 \leq \frac{2}{m+1} \|x\|_2^2 + \frac{4}{m+1} \sum_{l=1}^m |\widehat{\sigma_x}(l)|.$$



Since the measure  $\sigma_x$  is continuous by Proposition 3.2, Wiener's criterion of continuity of positive finite Borel measure ([5], p.42) yields

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{l=1}^m |\widehat{\sigma_x}(l)|^2 = 0,$$

which entails

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{l=1}^m |\widehat{\sigma_x}(l)| = 0.$$

Thus, we conclude that

$$(7) \quad \lim_{n \rightarrow \infty} \sup_{\lambda \in \mathbb{C}_1} \|a_n(x, \lambda)p\|_\infty = 0,$$

whence  $x \in WW$ . □

Note that (7) can be referred as non-commutative Bourgain's uniform Wiener-Wintner ergodic theorem.

**Remark 5.1.** As we have noticed (Proposition 2.2), for a fixed  $\lambda \in \mathbb{C}_1$  and every  $x \in \mathcal{L}^1$ , the averages  $a_n(x, \lambda)$  converge b.a.u. to some  $x_\lambda \in \mathcal{L}^1$ . It can be verified [7] that  $x_\lambda$  is a scalar multiple of  $\mathbb{I}$ . If we assume additionally that  $\alpha$  is *weakly mixing* in  $\mathcal{L}^2$ , that is 1 is its only eigenvalue there, then it is easy to see that the b.a.u. limit of  $\{a_n(x, \lambda)\}$  with  $x \in \mathcal{L}^2$  is zero unless  $\lambda = 1$ . Since  $\mathcal{L}^2$  is dense in  $\mathcal{L}^1$ , one can employ an argument similar to that of Theorem 2.1 to show that  $a_n(x, \lambda) \rightarrow 0$  b.a.u. for every  $x \in \mathcal{L}^1$  if  $\lambda \neq 1$ . Therefore if  $\alpha$  is weakly mixing, we can replace, in Theorem 5.1,

$$\{pa_n(x, \lambda)p\} \text{ converges in } \mathcal{M} \text{ for all } \lambda \in \mathbb{C}_1.$$

by

$$\|pa_n(x, \lambda)p\|_\infty \rightarrow 0 \text{ if } \lambda \neq 1 \text{ and } \|p(a_n(x) - x_1)p\|_\infty \rightarrow 0 \text{ for some } x_1 \in \mathcal{L}^1;$$

see Proposition 2.2 and Remark 2.1.

## REFERENCES

- [1] I. Assani, **Wiener Wintner ergodic theorems**, World Scientific (2003)
- [2] V. Chilin, S. Litvinov, *Uniform equicontinuity for sequences of homomorphisms into the ring of measurable operators*, Methods of Funct. Anal. Top., 12 (2) (2006), 124-130
- [3] V. Chilin, S. Litvinov, A. Skalski, *A few remarks in non-commutative ergodic theory*, J. Operator Theory, 53 (2) (2005), 331-350
- [4] M. Junge, Q. Xu, *Noncommutative maximal ergodic theorems*, J. Amer. Math. Soc., 20 (2)(2007), 385-439
- [5] Y. Katznelson, **An introduction to harmonic analysis**, Dover Publications (1976)
- [6] S. Litvinov, *Uniform equicontinuity of sequences of measurable operators and non-commutative ergodic theorems*, Proc. of Amer. Math. Soc., 140 (2012), 2401-2409
- [7] S. Litvinov, *Weighted ergodic theorems*, Doctoral Dissertation, North Dakota State University (1999)
- [8] E. Nelson, *Notes on non-commutative integration*, J. Funct. Anal., 15 (1974), 103-116
- [9] C. P. Niculescu, A. Ströh, L. Zsidó, *Noncommutative extensions of classical and multiple recurrence theorems*, J. Operator Theory, 50 (2005), 3-52
- [10] I. Segal, *A non-commutative extension of abstract integration*, Ann. of Math., 57 (1953), 401-457
- [11] F. J. Yeadon, *Non-commutative  $L^p$ -spaces*, Math. Proc. Camb. Philos. Soc., 77 (1975), 91-102

- [12] F. J. Yeadon, *Ergodic theorems for semifinite von Neumann algebras-I*, J. London Math. Soc., 16 (2) (1977), 326-332

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